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Kinetic Relaxation Models for Energy Transport

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Kinetic equations with relaxation collision kernels are considered under the basic assumption of two collision invariants, namely mass and energy. The collision kernels are of BGK-type with a general local Gibbs state, which may be quite different from the Gaussian. By the use of the diffusive length/time scales, energy transport systems consisting of two parabolic equations with the position density and the energy density as unknowns are derived on a formal level. The H theorem for the kinetic model is presented, and the entropy for the energy transport systems, which is inherited from the kinetic model, is derived. The energy transport systems for specific examples of the global Gibbs state, such as a power law with negative exponent, a cut-off power law with positive exponent, the Maxwellian, Bose–Einstein, and Fermi–Dirac distributions, are presented.

KEY WORDS: kinetic equation, diffusive limit, energy transport, Gibbs state, entropy

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1. INTRODUCTION

BGK models, named after Bhatnagar, Gross and Krook,[(1)] have taken a prominent role in the quest of simplifying collisional kinetic phase-space equations such as the Boltzmann equation for gas dynamics,[(2,3)] for current transport in semiconductors and plasmas,[(4)] for modeling transport of granular media,[(5)] etc. BGK models

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are based on the basic hypothesis that even large deviations from equilibrium in collisional flows can be described by the equilibrium momentum distribution when certain parameters are not taken as constants but made position and time dependent. Typically, the global momentum equilibrium is determined by the Gibbs state (function of the energy) with constant position space density, quasi-Fermi level and temperature. In the corresponding BGK model, these constants (or a subset of them) are replaced by functions of position and time, which are determined from physical conservation laws corresponding to so-called collision invariants,[(2)] and relaxation of the phase space distribution to the local Gibbs state is assumed. This makes the main advantage of BGK models apparent: they describe the collisional process 'well' in the sense that they feature the same equilibrium distribution and (some of) the same collision invariants without requiring specific information on collision details as present in the Boltzmann equation.

In the classical dynamics of ideal rarefied gases, there is conservation of mass, momentum and energy in each individual collision of two gas atoms or molecules, which, at least on the formal level, carries over to the whole ensemble of gas particles, when its dynamics is represented by the Boltzmann equation. The equilibrium momentum distribution is the Gaussian (in momentum space), with five parameters (in six dimensional phase space!) represented by the position density, the mean velocity vector (three dimensional parameter function) and temperature. The corresponding classical BGK model[(1)] determines these three parameters as function of space-time by computing them directly from the time-dependent position-momentum distribution function.

Simpler BGK models can be set up. First of all, not all (five) collision invariants need to be taken into account. For example, in semiconductor charge transport theory, momentum and energy are transferred to the crystal lattice and thus not conserved by the charge carrier ensembles. The corresponding Gaussian BGK model turns out to be linear in the charge carrier distribution function.[(4)]

Another possibility is to prescribe a momentum equilibrium distribution different from the Gaussian. We refer to Ref. 6, where BGK models with a general Gibbs state and only one conservative quantity (mass) were analyzed.

In this paper we consider BGK models with a general basic Gibbs state of zero mean velocity, which may very well be different from the Gaussian, under the basic assumption of two conservative quantities, namely mass and energy. Thus we consider generalized energy transport models, NOT based on a Maxwellian energy distribution. We refer to Ref. 7 for the Maxwellian case. In particular we shall scale the kinetic BGK models by using diffusive length/time scales and, on a formal level, consider the diffusive limit, always in dependence of the general Gibbs state equilibrium. Generalizing results of Ref. 7 we obtain energy transport systems consisting of two parabolic equations with the position density and the energy density as unknowns. Also, we discuss stationary states, and derive an entropy for the parabolic system, which is 'inherited' directly from the kinetic problem.

In Ref. 6 the diffusive limit from a one-parameter BGK model (with mass conservation) with general Gibbs state to a generally nonlinear diffusion equation was analyzed and convergence was proved under certain hypothesis on the Gibbs state etc. The computations in this paper, in particular the asymptotics leading to the generalized energy transport system from the two-parameter BGK kinetic model (mass and energy conservation) with the general Gibbs state, are purely formal. We expect a proof of convergence to be much more difficult than in the already very involved one parameter case.

Energy transport models are very useful in certain physical applications, particularly when thermal convection is important when nonlinear effects due to kinematic convection are not of significant size. And an important application occurs in semiconductor physics, see e.g. Ref. 7, where non-Maxwellian energy transport systems, as derived in this paper, have big relevance, since charge carrier (i.e., Fermion) equilibria obey the Fermi-Dirac distribution (see Example 4.4).

2. SCALED KINETIC EQUATION AND ITS FORMAL MACROSCOPIC LIMIT

2.1. Formulation and Conservation Laws

We consider the scaled kinetic equation

$$\varepsilon^2 \partial_t f + \varepsilon[v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f] = G_f - f, \tag{1}$$

$$G_f := \gamma(E_f), \quad E_f := \left(\frac{1}{2}|v|^2 - \mu(\rho_f, e_f)\right) / \theta(\rho_f, e_f), \tag{2}$$

where the particle velocity distribution function $f = f(x, v, t)$ depends on position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$, and time $t > 0$. The external potential $V(x)$ is given. The collision model is a simple relaxation kernel toward a generalized local Gibbs state G_f. The chemical potential $\mu(\rho_f, e_f)$ (the Gibbs free energy per unit mass) and the *temperature* $\theta(\rho_f, e_f) > 0$ are to be determined implicitly by the conditions

$$\int_{\mathbb{R}^3} G_f dv = \rho_f(x, t) := \int_{\mathbb{R}^3} f dv, \tag{3}$$

$$\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 G_f dv = e_f(x, t) := \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 f dv, \tag{4}$$

or equivalently

$$\rho_f(x, t) = \int_{\mathbb{R}^3} \gamma\left(\left[\frac{1}{2}|v|^2 - \mu(\rho_f, e_f)\right] \Big/ \theta(\rho_f, e_f)\right) dv, \tag{5}$$

$$e_f(x, t) = \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 \gamma\left(\left[\frac{1}{2}|v|^2 - \mu(\rho_f, e_f)\right] \Big/ \theta(\rho_f, e_f)\right) dv, \tag{6}$$

where ρ_f is the (macroscopic) density and e_f is the internal energy per unit volume. It should be noted that the pair of variables (μ, θ) depends on x and t only through (ρ_f, e_f) and vice versa. The local invertibility of the map from (μ, θ) to (ρ_f, e_f) holds (see below) while the global invertibility will be assumed in the present work. The quantity $\theta(\rho_f, e_f)E_f$ may be interpreted as the energy of a particle.

Equation (1) is considered subject to initial conditions

$$f(x, v, 0) = f_I(x, v), \tag{7}$$

with $f_I \in L^1_+(\mathbb{R}^3 \times \mathbb{R}^3)$.

If we put $\theta = 1$ in (2) and discard the condition (4) [and (6)], we have the relaxation kinetic model for the mass transport. The existence and uniqueness of solutions of this model and its diffusive macroscopic limit have already been studied rigorously mathematically by Dolbeault, Markowich, Oelz and Schmeiser.[6] In the present paper, we shall extend this result to the mass and energy transports at the formal level.

In the above formulation, we have used the quantities conventional in thermodynamics and fluid dynamics. For later discussions, however, it would be convenient to use the quasi Fermi potential $\bar{\mu}$ and the energy $\mathcal{E}_f$ per unit volume

$$\mathcal{E}_f(x, t) = e_f(x, t) + \rho_f(x, t)V(x), \quad \bar{\mu}(x, t) = (\mu(x, t) + V(x))/\theta(x, t),$$

in place of e_f and μ. Then $(\bar{\mu}, \theta)$ is considered to depend on x and t through $(\rho_f, \mathcal{E}_f, V)$ and $(\rho_f, \mathcal{E}_f)$ does through $(\bar{\mu}, \theta, V)$, and E_f is rewritten as

$$E_f(\rho_f, \mathcal{E}_f, V) = \left(\frac{1}{2}|v|^2 + V(x)\right) / \theta(\rho_f, \mathcal{E}_f - \rho_f V) - \bar{\mu}(\rho_f, \mathcal{E}_f, V).$$

Because of the conditions (3) and (4), integrating Eq. (1) multiplied by 1 and $\frac{1}{2}|v|^2 + V(x)$ over the whole space of v yields the continuity and energy transport equations:

$$\varepsilon \partial_t \rho_f + \nabla_x \cdot \int_{\mathbb{R}^3} v f dv = 0, \tag{8}$$

$$\varepsilon \partial_t \mathcal{E}_f + \nabla_x \cdot \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 + V(x)\right) v f dv = 0, \tag{9}$$

provided that f decays sufficiently fast as $|v| \to \infty$. Integrations of these equations with respect to x over $\mathbb{R}^3$ show that the total mass $M := \int_{\mathbb{R}^6} f_I(x, v)\, dx dv$ and the total energy $U := \int_{\mathbb{R}^6} (\frac{1}{2}|v|^2 + V(x)) f_I(x, v)\, dx dv$ are preserved by the time evolution, i.e., $\int_{\mathbb{R}^6} f(x, v, t)\, dx dv = M$ and $\int_{\mathbb{R}^6} (\frac{1}{2}|v|^2 + V(x)) f(x, v, t)\, dx dv = U$ for all $t > 0$, as far as f decays sufficiently fast as $|x| \to \infty$.

In the present paper, we carry out formal analyses by assuming that

(i) The Gibbs state $\gamma(E)$ is a nonincreasing and nonnegative continuous function of E in the interval (E_1, ∞), where E_1 is a constant including $-\infty$.

The function γ has a support $[E_1, E_2]$ and is continuously differentiable on (E_1, E_2) with $\dot{\gamma}(E) < 0$, where E_2 is a constant including ∞. Thus it has an inverse γ^{-1} mapping into (E_1, E_2). Since γ is nonnegative, both the density ρ_f and the internal energy e_f are also nonnegative.

(ii) In the case where $E_2 = \infty$, there exist a $\delta > 0$ such that $\gamma(E) = O(E^{-\frac{7}{2}-\delta})$ as $E \to \infty$. This assumption ensures the existence of fourth order velocity moments, say a_2 and A_2, which shall appear later in the analysis.

(iii) The external potential V is bounded from below and thus may be chosen nonnegative. Consequently the energy $\mathcal{E}_f$ is also nonnegative.

(iv) The map (5) and (6) between the pairs (ρ_f, e_f) and (μ, θ) is invertible.

As to the last assumption, we remark that the map (5) and (6) is locally invertible. In fact, a straightforward calculation shows that the Jacobian is written as

$$\frac{\partial(\rho_f, e_f)}{\partial(\mu, \theta)} = \frac{1}{\theta^3}\left(\int_{\mathbb{R}^3} \dot{\gamma}(E_f)\,dv \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2\right)^2 \dot{\gamma}(E_f)\,dv - \left(\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 \dot{\gamma}(E_f)\,dv\right)^2\right),$$

the right-hand side of which is positive because of the assumption (i) and the Cauchy–Schwarz inequality (with the corresponding condition for equality). Therefore, the Jacobian is non-zero and local invertibility, say around an appropriately chosen initial state, is assured by the inverse function theorem.

2.2. Formal Macroscopic Limit

Consider formal asymptotic expansions $f = f^0 + \varepsilon f^1 + O(\varepsilon^2)$, $\bar{\mu} = \bar{\mu}^0 + O(\varepsilon)$, $\theta = \theta^0 + O(\varepsilon)$, $\rho_f = \rho^0 + O(\varepsilon)$, and $\mathcal{E}_f = \mathcal{E}^0 + O(\varepsilon)$. Then, by going to the limit $\varepsilon \to 0$ in (1), we obtain, at the lowest order in ε,

$$f^0(x, v, t) = G^0(x, v, t) = \gamma(E^0),$$

where

$$E^0 = \left(\frac{1}{2}|v|^2 + V(x)\right)/\theta^0(x, t) - \bar{\mu}^0(x, t),$$

with

$$\bar{\mu}^0(x, t) = \bar{\mu}(\rho^0, \mathcal{E}^0, V), \quad \theta^0(x, t) = \theta(\rho^0, \mathcal{E}^0 - \rho^0 V),$$

and

$$\rho^0(x, t) = \int_{\mathbb{R}^3} f^0 dv, \quad \mathcal{E}^0(x, t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 + V(x)\right) f^0 dv.$$

The $O(\varepsilon)$-terms in (1) give

$$v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1,$$

which can be rewritten as

$$f^1 = G^1 - \dot{\gamma}(E^0)\, v \cdot \nabla_x E^0 + \dot{\gamma}(E^0)\, v \cdot \frac{\nabla_x V}{\theta^0}. \tag{10}$$

Now we pass to the limit in Eqs. (8) and (9) and obtain

$$\partial_t \rho^0 + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 dv = 0,$$

$$\partial_t \mathcal{E}^0 + \nabla_x \cdot \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 + V(x)\right) v f^1 dv = 0.$$

For the evaluation of the fluxes, we use (10). Because of the form of E_f, G^1 is an even function of v. Therefore it does not contribute, and we end up with

$$\partial_t \rho^0 - \frac{1}{3}\nabla_x \cdot \left(\nabla_x \int_{\mathbb{R}^3} |v|^2 \gamma(E^0)\, dv - \int_{\mathbb{R}^3} |v|^2 \dot{\gamma}(E^0)\, dv\, \frac{\nabla_x V}{\theta^0}\right) = 0,$$

$$\partial_t \mathcal{E}^0 - \frac{1}{3}\nabla_x \cdot \left(\nabla_x \int_{\mathbb{R}^3} |v|^2 \left(\frac{1}{2}|v|^2 + V\right) \gamma(E^0)\, dv \right.$$

$$\left. - \int_{\mathbb{R}^3} |v|^2 \left[\left(\frac{1}{2}|v|^2 + V\right) \frac{\dot{\gamma}(E^0)}{\theta^0} + \gamma(E^0)\right] dv\, \nabla_x V\right) = 0,$$

or equivalently,

$$\partial_t \rho^0 \;\; -\frac{1}{3}\nabla_x \cdot \left(2\nabla_x(\mathcal{E}^0 - \rho^0 V) + \frac{a_1}{\theta^0}\nabla_x V\right) = 0, \tag{11}$$

$$\partial_t \mathcal{E}^0 \;\; -\frac{1}{3}\nabla_x \cdot \left(\nabla_x \bar{A}_1 + \left[\frac{\bar{a}_1}{\theta^0} - 2(\mathcal{E}^0 - \rho^0 V)\right] \nabla_x V\right) = 0, \tag{12}$$

where

$$a_n := -\int_{\mathbb{R}^3} |v|^{2n} \dot{\gamma}(E^0)\, dv = -4\pi \int_0^\infty |v|^{2(n+1)} \dot{\gamma}(E^0)\, d|v|,$$

$$\bar{a}_n := -\int_{\mathbb{R}^3} |v|^{2n} \left(\frac{1}{2}|v|^2 + V(x)\right) \dot{\gamma}(E^0)\, dv = a_n V + \frac{1}{2} a_{n+1},$$

$$A_n := \int_{\mathbb{R}^3} |v|^{2n} \gamma(E^0)\, dv = 4\pi \int_0^\infty |v|^{2(n+1)} \gamma(E^0)\, d|v|,$$

$$\bar{A}_n := \int_{\mathbb{R}^3} |v|^{2n} \left(\frac{1}{2}|v|^2 + V(x)\right) \gamma(E^0)\, dv = A_n V + \frac{1}{2} A_{n+1}.$$

Here we have used the fact that E^0 is a function of $|v|^2$ rather than v itself, and so are $\gamma(E^0)$ and $\dot\gamma(E^0)$. Note that $A_0 = \rho^0$ and $A_1 = 2(\mathcal{E}^0 - \rho^0 V)$ by definition. Equations (11) and (12) are the diffusion equations in conservative formulation. The initial data of these equations are given by

$$\rho^0(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv,$$

$$\mathcal{E}^0(x,0) = \mathcal{E}_I(x) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|v|^2 + V(x)\right) f_I(x,v)\,dv,$$

because both the density ρ_f and the energy $\mathcal{E}_f$ are preserved in the initial layer governed by $\partial_\tau f = G_f - f$ with $\tau = t/\varepsilon^2$.

The diffusion equations (11) and (12) are in conservative form but are not in symmetric one. In order to derive the symmetric formulation, we rewrite Eq. (10) as

$$f^1 = G^1 - \dot\gamma(E^0)\, v \cdot \left(\left(\frac{1}{2}|v|^2 + V\right)\nabla_x \frac{1}{\theta^0} - \nabla_x \bar\mu^0\right), \tag{13}$$

and use this expression for the evaluation of the fluxes. Then we obtain

$$\bar a_0 \partial_t \frac{1}{\theta^0} - a_0 \partial_t \bar\mu^0 - \frac{1}{3}\nabla_x \cdot \left(\bar a_1 \nabla_x \frac{1}{\theta^0} - a_1 \nabla_x \bar\mu^0\right) = 0, \tag{14}$$

$$\left(\frac{1}{2}\bar a_1 + \bar a_0 V\right)\partial_t \frac{1}{\theta^0} - \bar a_0 \partial_t \bar\mu^0$$
$$-\frac{1}{3}\nabla_x \cdot \left(\left(\frac{1}{2}\bar a_2 + \bar a_1 V\right)\nabla_x \frac{1}{\theta^0} - \bar a_1 \nabla_x \bar\mu^0\right) = 0. \tag{15}$$

Here we have used the relations

$$\partial_t \rho^0 = -\bar a_0 \partial_t \frac{1}{\theta^0} + a_0 \partial_t \bar\mu^0, \quad \partial_t \mathcal{E}^0 = -\left(\frac{1}{2}\bar a_1 + \bar a_0 V\right)\partial_t \frac{1}{\theta^0} + \bar a_0 \partial_t \bar\mu^0.$$

Equations (14) and (15) can be written, in terms of the new notation $\varphi_1 := -\bar\mu^0$ and $\varphi_2 := 1/\theta^0$, as

$$a_0 \partial_t \varphi_1 + \bar a_0 \partial_t \varphi_2 - \frac{1}{3}\nabla_x \cdot (a_1 \nabla_x \varphi_1 + \bar a_1 \nabla_x \varphi_2) = 0,$$

$$\bar a_0 \partial_t \varphi_1 + \left(\frac{1}{2}\bar a_1 + \bar a_0 V\right)\partial_t \varphi_2$$
$$-\frac{1}{3}\nabla_x \cdot \left(\bar a_1 \nabla_x \varphi_1 + \left(\frac{1}{2}\bar a_2 + \bar a_1 V\right)\nabla_x \varphi_2\right) = 0,$$

or equivalently

$$\sum_{j=1}^{2} A_{ij}\partial_t\varphi_j - \frac{1}{3}\sum_{\ell,m=1}^{3}\partial_{x_\ell}\sum_{j=1}^{2} D_{ij}^{\ell m}\partial_{x_m}\varphi_j = 0, \qquad i = 1, 2, \tag{16}$$

with

$$A_{ij} := a_0\delta_{i1}\delta_{j1} + \bar{a}_0(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) + \left(\bar{a}_0 V + \frac{1}{2}\bar{a}_1\right)\delta_{i2}\delta_{j2},$$

$$D_{ij}^{\ell m} := \left(a_1\delta_{i1}\delta_{j1} + \bar{a}_1(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) + \left(\bar{a}_1 V + \frac{1}{2}\bar{a}_2\right)\delta_{i2}\delta_{j2}\right)\delta_{\ell m}.$$

It is obvious that the diffusion equation (16) is symmetric: $A_{ij} = A_{ji}$ and $D_{ij}^{\ell m} = D_{ji}^{m\ell}$. Further, the tensors $A = (A_{ij})$ and $D = (D_{ij}^{\ell m})$ are both positive definite, which is shown below:

Proposition 2.1. *The tensors A and D occurring in Eq. (16) are positive definite.*

Proof: We first show that the tensor A is positive definite. For any nonzero vector x_i $(i = 1, 2)$,

$$\sum_{i,j=1}^{2} A_{ij}x_i x_j = \sum_{i,j=1}^{2}\left[a_0\delta_{i1}\delta_{j1} + \bar{a}_0(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) + \left(\bar{a}_0 V + \frac{1}{2}\bar{a}_1\right)\delta_{i2}\delta_{j2}\right]x_i x_j$$

$$= a_0 x_1^2 + 2\bar{a}_0 x_1 x_2 + \left(\bar{a}_0 V + \frac{1}{2}\bar{a}_1\right)x_2^2$$

$$= a_0\left(x_1 + \frac{\bar{a}_0}{a_0}x_2\right)^2 + \frac{1}{a_0}\left(a_0\left(\bar{a}_0 V + \frac{1}{2}\bar{a}_1\right) - \bar{a}_0^2\right)x_2^2.$$

Since $\dot{\gamma}(E^0) \le 0$ by assumption (i) in Sec. 2.1, the Cauchy–Schwarz inequality leads to

$$a_0\left(\bar{a}_0 V + \frac{1}{2}\bar{a}_1\right) = \int_{\mathbb{R}^3}\dot{\gamma}(E^0)\,dv\int_{\mathbb{R}^3}\left(\frac{1}{2}|v|^2 + V\right)^2\dot{\gamma}(E^0)\,dv$$

$$> \left(\int_{\mathbb{R}^3}\left(\frac{1}{2}|v|^2 + V\right)\dot{\gamma}(E^0)\,dv\right)^2 = \bar{a}_0^2.$$

Therefore $\sum_{i,j=1}^{2} A_{ij}x_i x_j > 0$ because a_0 is positive.

In the same way, we can show the tensor D to be positive definite: for any nonzero tensor T_i^ℓ ($i = 1, 2$, $\ell = 1, 2, 3$),

$$\begin{aligned}
&\sum_{\ell,m=1}^{3}\sum_{i,j=1}^{2} D_{ij}^{\ell m} T_i^\ell T_j^m \\
&= \sum_{\ell,m=1}^{3}\sum_{i,j=1}^{2}\left(a_1\delta_{i1}\delta_{j1}+\bar{a}_1(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}) + \left(\bar{a}_1 V + \frac{1}{2}\bar{a}_2\right)\delta_{i2}\delta_{j2}\right)\delta_{\ell m} T_i^\ell T_j^m \\
&= \sum_{\ell=1}^{3}\left(a_1\left(T_1^\ell\right)^2 + 2\bar{a}_1 T_1^\ell T_2^\ell + \left(\bar{a}_1 V + \frac{1}{2}\bar{a}_2\right)\left(T_2^\ell\right)^2\right) \\
&= \sum_{\ell=1}^{3}\left(a_1\left(T_1^\ell + \frac{\bar{a}_1}{a_1}T_2^\ell\right)^2 + \frac{1}{a_1}\left(a_1\left(\bar{a}_1 V + \frac{1}{2}\bar{a}_2\right) - \bar{a}_1^2\right)\left(T_2^\ell\right)^2\right).
\end{aligned}$$

Again, because of the Cauchy–Schwarz inequality,

$$\begin{aligned}
a_1\left(\bar{a}_1 V + \frac{1}{2}\bar{a}_2\right) &= \int_{\mathbb{R}^3} |v|^2 \dot{\gamma}(E^0)\,dv \int_{\mathbb{R}^3} |v|^2 \left(\frac{1}{2}|v|^2 + V\right)^2 \dot{\gamma}(E^0)\,dv \\
&> \left(\int_{\mathbb{R}^3} |v|^2 \left(\frac{1}{2}|v|^2 + V\right)\dot{\gamma}(E^0)\,dv\right)^2 = \bar{a}_1^2,
\end{aligned}$$

and this concludes $\sum_{\ell,m=1}^{3}\sum_{i,j=1}^{2} D_{ij}^{\ell m} T_i^\ell T_j^m > 0$ because a_1 is positive. □

3. STEADY SOLUTIONS AND ENTROPIES OF THE KINETIC AND MACROSCOPIC EQUATIONS

It is readily seen by substitution that the Gibbs state with constant θ and $\bar{\mu}$ is a steady solution of the kinetic equation (1) and correspondingly that $\varphi_i = \text{const}$ ($i = 1, 2$) is a steady solution of the diffusion equation (16). Although these facts do not exclude the possibility of other steady solutions, actually they are not allowed, as will be shown in the subsequent subsections.

3.1. H Theorem for the Kinetic Equation

Consider the function

$$H(x, t) := -\int_{\mathbb{R}^3}\left(\int_0^{f(x,v,t)} \gamma^{-1}(s)\,ds\right)dv,$$

where f solves the kinetic equation (1). The time derivative of H is calculated with the aid of (1) as follows:

$$
\begin{aligned}
&\varepsilon^2 \partial_t H(x,t)\\
&\quad = -\int_{\mathbb{R}^3} \gamma^{-1}(f)\varepsilon^2 \partial_t f\, dv\\
&\quad = \varepsilon \int_{\mathbb{R}^3} \gamma^{-1}(f)(v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f)\, dv + \int_{\mathbb{R}^3} \gamma^{-1}(f)(f-\gamma(E_f))\, dv\\
&\quad = \varepsilon \int_{\mathbb{R}^3} \left(v\cdot\nabla_x \int_0^f \gamma^{-1}(s)\, ds - \nabla_x V\cdot\nabla_v \int_0^f \gamma^{-1}(s)\, ds \right) dv\\
&\qquad + \int_{\mathbb{R}^3} (\gamma^{-1}(f) - E_f)(f-\gamma(E_f))\, dv\\
&\quad = \varepsilon \nabla_x \cdot \int_{\mathbb{R}^3} v \left(\int_0^f \gamma^{-1}(s)\, ds \right) dv + \int_{E_f<E_2} (\gamma^{-1}(f) - \gamma^{-1}(\gamma(E_f)))\\
&\qquad \times (f-\gamma(E_f))\, dv + \int_{E_f>E_2} (\gamma^{-1}(f) - E_f) f\, dv\\
&\quad = -\varepsilon \nabla_x \cdot H_{\text{flux}} - D(f),
\end{aligned}
$$

where

$$
\begin{aligned}
H_{\text{flux}} &:= -\int_{\mathbb{R}^3} \left(\int_0^{f(x,v,t)} \gamma^{-1}(s)\, ds \right) v\, dv,\\
D(f) &:= -\int_{E_f<E_2} (\gamma^{-1}(f) - \gamma^{-1}(\gamma(E_f)))(f-\gamma(E_f))\, dv\\
&\quad - \int_{E_f>E_2} \left(\gamma^{-1}(f) - E_f \right) f\, dv\\
&= -\int_{E_f<E_2} \frac{\gamma^{-1}(f) - \gamma^{-1}(\gamma(E_f))}{f-\gamma(E_f)} (f-\gamma(E_f))^2\, dv\\
&\quad - \int_{E_f>E_2} \left(\gamma^{-1}(f) - E_f \right) f\, dv.
\end{aligned}
$$

Since γ^{-1} is decreasing for $E_f < E_2$ by assumption, the integrand in the first term in the last equation is nonpositive because of the mean value theorem. In addition, the integrand in the second term is also nonpositive because $\gamma^{-1} < E_2$. Therefore, the last expression of $D(f)$ implies $D(f) \geq 0$, and the equality holds if and only if f is the Gibbs state, i.e., $f = \gamma(E_f)$. Therefore, the function H satisfies the inequality

$$\varepsilon^2 \partial_t H + \varepsilon \nabla_x \cdot H_{\text{flux}} = -D(f) \le 0, \tag{17}$$

and the equality holds if and only if f is the Gibbs state. This means that H is no other than the *H function* for the kinetic equation (1).

Equation (17) leads to the inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^3} H dx \le 0, \tag{18}$$

where the condition for the equality is the same as before. Therefore in the steady state, f must be the Gibbs state, i.e., $f = \gamma(E_f)$. By substituting the Gibbs state in Eq. (1) without the time derivative term, we see that θ and $\bar{\mu}$ must be constant in the steady state:

Proposition 3.1. *The steady solution of the kinetic equation (1) is a Gibbs state with constant θ and $\bar{\mu}$.*

3.2. Entropy Associated to the Diffusion Equation

Consider the function

$$S(x,t) := -\int_{\mathbb{R}^3} \left(\int_0^{\gamma(E^0)} \gamma^{-1}(s)\, ds \right) dv,$$

with $E^0 = \varphi_1 + (\frac{1}{2}|v|^2 + V)\varphi_2$, where φ_i's solve the diffusion equation (16). Then the time derivative of S is calculated with the aid of (16) as follows:

$$\begin{aligned}
&\partial_t S(x,t) \\
&= -\int_{\mathbb{R}^3} \partial_t \gamma(E^0) \gamma^{-1}(\gamma(E^0))\, dv \\
&= -\int_{\mathbb{R}^3} \gamma^{-1}(\gamma(E^0)) \dot{\gamma}(E^0) \left(\partial_t \varphi_1 + \left(\frac{1}{2}|v|^2 + V \right) \partial_t \varphi_2 \right) dv \\
&= -\int_{E^0 < E_2} E^0 \dot{\gamma}(E^0) \left(\partial_t \varphi_1 + \left(\frac{1}{2}|v|^2 + V \right) \partial_t \varphi_2 \right) dv \\
&= -\int_{\mathbb{R}^3} \left(\left(\frac{1}{2}|v|^2 + V \right) \varphi_2 + \varphi_1 \right) \dot{\gamma}(E^0) \left(\partial_t \varphi_1 + \left(\frac{1}{2}|v|^2 + V \right) \partial_t \varphi_2 \right) dv \\
&= \varphi_2 \left(\bar{a}_0 \partial_t \varphi_1 + \left(\frac{1}{2}\bar{a}_1 + \bar{a}_0 V \right) \partial_t \varphi_2 \right) + \varphi_1 (a_0 \partial_t \varphi_1 + \bar{a}_0 \partial_t \varphi_2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\varphi_2 \nabla_x \cdot \left(\bar{a}_1 \nabla_x \varphi_1 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) \nabla_x \varphi_2\right) + \frac{1}{3}\varphi_1 \nabla_x \cdot (a_1 \nabla_x \varphi_1 + \bar{a}_1 \nabla_x \varphi_2) \\
&= \frac{1}{3}\nabla_x \cdot \left[\varphi_2 \left(\bar{a}_1 \nabla_x \varphi_1 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) \nabla_x \varphi_2\right) + \varphi_1 (a_1 \nabla_x \varphi_1 + \bar{a}_1 \nabla_x \varphi_2)\right] \\
&\quad - \frac{1}{3}\nabla_x \varphi_2 \cdot \left(\bar{a}_1 \nabla_x \varphi_1 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) \nabla_x \varphi_2\right) - \frac{1}{3}\nabla_x \varphi_1 \cdot (a_1 \nabla_x \varphi_1 + \bar{a}_1 \nabla_x \varphi_2) \\
&= \frac{1}{3}\nabla_x \cdot \left[\varphi_2 \left(\bar{a}_1 \nabla_x \varphi_1 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) \nabla_x \varphi_2\right) + \varphi_1 (a_1 \nabla_x \varphi_1 + \bar{a}_1 \nabla_x \varphi_2)\right] \\
&\quad - \frac{1}{3}\left(a_1 |\nabla_x \varphi_1|^2 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) |\nabla_x \varphi_2|^2 + 2\bar{a}_1 \nabla_x \varphi_1 \cdot \nabla_x \varphi_2\right) \\
&= -\nabla_x \cdot S_{\text{flux}} - DS,
\end{aligned}
$$

where

$$
\begin{aligned}
S_{\text{flux}} &:= -\frac{1}{3}\left[\varphi_2 \left(\bar{a}_1 \nabla_x \varphi_1 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) \nabla_x \varphi_2\right) + \varphi_1 (a_1 \nabla_x \varphi_1 + \bar{a}_1 \nabla_x \varphi_2)\right], \\
DS &:= \frac{1}{3}\left(a_1 |\nabla_x \varphi_1|^2 + \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) |\nabla_x \varphi_2|^2 + 2\bar{a}_1 \nabla_x \varphi_1 \cdot \nabla_x \varphi_2\right) \\
&= \frac{1}{3}\left(a_1 \left(\nabla_x \varphi_1 + \frac{\bar{a}_1}{a_1}\nabla_x \varphi_2\right)^2 + \frac{1}{a_1}\left(a_1 \left(\frac{1}{2}\bar{a}_2 + \bar{a}_1 V\right) - \bar{a}_1^2\right) |\nabla_x \varphi_2|^2\right).
\end{aligned}
$$

The last expression of DS implies $DS \geq 0$ because of the Cauchy–Schwarz inequality, and the equality holds if and only if φ_i's are constant in x. Therefore, the function S satisfies the *entropy inequality*

$$
\partial_t S + \nabla_x \cdot S_{\text{flux}} = -DS \leq 0, \tag{19}
$$

where the equality holds if and only if φ_i's are constant in x. We call S the *entropy*, S_{flux} the *entropy flux*, and DS the *entropy dissipation* associated to the diffusion equation (16).

The entropy inequality (19) leads to the inequality for the *total entropy* $\int_{\mathbb{R}^3} S dx$:

$$
\frac{d}{dt}\int_{\mathbb{R}^3} S dx \leq 0, \tag{20}
$$

where the condition for the equality is the same as before. Therefore,

Proposition 3.2. *The steady solution of the diffusion equation (16) is* $\varphi_i = \text{const}$ *(*$i = 1, 2$*).*

At this point, we refer to the study of the symmetrization of diffusion equations and their associated entropy by Degond, Génieys, and Jüngel.[8] The set of the diffusion equations (11) and (12) falls into the class of equations discussed in this reference when V is zero. Although our discussion on the entropy S so far is based on the symmetric formulation, S may be considered as the entropy associated to Eqs. (11) and (12) and can be written as

$$S(x,t) = -\int_{\mathbb{R}^3} \left(\int_0^{\gamma(E^0)} \gamma^{-1}(s)\, ds \right) dv,$$

with $E^0 = [\frac{1}{2}|v|^2 - \mu(\rho^0, \mathcal{E}^0)]/\theta(\rho^0, \mathcal{E}^0)$. Then it can be shown that this S satisfies the definition 2.1 of the *associated entropy* in Ref. 8; and, consequently, the theorem 2.3 there assures that the change of variables from $(\rho^0, \mathcal{E}^0)$ to $(\partial S/\partial\rho^0, \partial S/\partial\mathcal{E}^0)$, i.e., the transformation to the *entropic variables*, symmetrizes the diffusion equations (11) and (12) with $V = 0$. In fact, the direct calculation shows $\partial S/\partial\rho^0 = \mu^0/\theta^0$ and $\partial S/\partial\mathcal{E}^0 = -1/\theta^0$ with $\mu^0 = \mu(\rho^0, \mathcal{E}^0)$, i.e., $\partial S/\partial\rho^0 = -\varphi_1$ and $\partial S/\partial\mathcal{E}^0 = -\varphi_2$. Therefore, the variables φ_1 and φ_2 introduced in Sec. 2.2 are essentially the same as the *entropic variables* in Ref. 8 when $V = 0$. It is also easily seen that $\rho^0 = \partial S^*/\partial\varphi_1$ and $\mathcal{E}^0 = \partial S^*/\partial\varphi_2$, where $S^* = S + \rho^0\varphi_1 + \mathcal{E}^0\varphi_2$ is the Legendre transformation of S.

4. EXAMPLES

We shall give some examples of the Gibbs state and the corresponding diffusion equations. As is obvious from the definition of the tensors A and D, the symmetric diffusion equation (16) is necessarily explicit, while the diffusion equations (11) and (12) for the density ρ^0 and the energy $\mathcal{E}^0$ are not always. In each example, we start with the former and turn to the latter.

Example 4.1. A power law with negative exponent, $\gamma(E) = CE^{-k}$, with $k > 7/2$ and C being a positive constant. The chemical potential is assumed nonpositive ($\mu \le 0$), so that $E \ge 0$.

We first derive the coefficients a_n's ($n = 0, 1, 2, 3$).

$$\begin{aligned} a_n &:= -\int_{\mathbb{R}^3} |v|^{2n} \dot\gamma(E^0)\, dv \\ &= Ck \int_{\mathbb{R}^3} |v|^{2n} (E^0)^{-k-1} dv \\ &= 4\pi Ck \int_0^\infty |v|^{2(n+1)} (\varphi_1 + \varphi_2 V)^{-k-1} \left(1 + \frac{|v|^2}{2} \frac{\varphi_2}{\varphi_1 + \varphi_2 V}\right)^{-k-1} d|v| \end{aligned}$$

$$= 2^{n+\frac{5}{2}}\pi Ck(\varphi_1+\varphi_2 V)^{n-k+\frac{1}{2}}\varphi_2^{-n-\frac{3}{2}}\int_0^\infty s^{n+\frac{1}{2}}(1+s)^{-k-1}ds$$

$$= 2^{n+\frac{5}{2}}\pi Ck(\varphi_1+\varphi_2 V)^{n-k+\frac{1}{2}}\varphi_2^{-n-\frac{3}{2}} B\left(n+\frac{3}{2}, k-n-\frac{1}{2}\right),$$

where B is the Beta function (the Euler integral of the first kind)[9] defined by

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt, \quad \Re x > 0, \quad \Re y > 0.$$

Here, from the third line to the fourth, we have put $s = \frac{|v|^2}{2}\frac{\varphi_2}{\varphi_1+\varphi_2 V}$ and have used the fact that s increases with $|v|$ increasing because of $\varphi_2 > 0$ and $E^0 > 0$. With this result and the identity[9]

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re x > 0, \quad \Re y > 0,$$

with Γ being the Gamma function, the tensors A and D in the symmetric diffusion equation (16) are finally written as

$$\begin{aligned}
A_{ij} = &\frac{8}{3}\sqrt{2}\pi CB\left(\frac{5}{2}, k-\frac{5}{2}\right)(\varphi_1+\varphi_2 V)^{-k+\frac{1}{2}}\varphi_2^{-\frac{7}{2}}\left[\left(k-\frac{3}{2}\right)\left(k-\frac{5}{2}\right)\varphi_2^2\delta_{i1}\delta_{j1}\right.\\
&+\left(k-\frac{5}{2}\right)\left(k\varphi_2 V+\frac{3}{2}\varphi_1\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})\\
&\left.+\left(k(k-1)(\varphi_2 V)^2+3k\varphi_1\varphi_2 V+\frac{15}{4}\varphi_1^2\right)\delta_{i2}\delta_{j2}\right],
\end{aligned}$$

$$\begin{aligned}
D_{ij}^{\ell m} = &\frac{16}{5}\sqrt{2}\pi CB\left(\frac{7}{2}, k-\frac{7}{2}\right)(\varphi_1+\varphi_2 V)^{-k+\frac{3}{2}}\varphi_2^{-\frac{9}{2}}\left[\left(k-\frac{5}{2}\right)\left(k-\frac{7}{2}\right)\varphi_2^2\delta_{i1}\delta_{j1}\right.\\
&+\left(k-\frac{7}{2}\right)\left(k\varphi_2 V+\frac{5}{2}\varphi_1\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})\\
&\left.+\left(k(k-1)(\varphi_2 V)^2+5k\varphi_1\varphi_2 V+\frac{35}{4}\varphi_1^2\right)\delta_{i2}\delta_{j2}\right]\delta_{\ell m}.
\end{aligned}$$

In the same way, the coefficients A_n's ($n = 0, 1, 2$) can be calculated as

$$A_n = 2^{n+\frac{5}{2}}\pi CB\left(n+\frac{3}{2}, k-n-\frac{3}{2}\right)(\varphi_1+\varphi_2 V)^{n-k+\frac{3}{2}}\varphi_2^{-n-\frac{3}{2}},$$

and, therefore, are related to a_n's as

$$A_n = \frac{\varphi_2}{2n+3} a_{n+1}.$$

Note that, by definition, $A_0 = \rho^0$, $A_1 = 2(\mathcal{E}^0 - \rho^0 V)$, $\varphi_1 + \varphi_2 V = -\mu^0/\theta^0$, and $\varphi_2 = 1/\theta^0$, where $\mu^0 = \mu(\rho^0, \mathcal{E}^0 - \rho^0 V)$. These identities lead to the explicit relation between (μ^0, θ^0) and (ρ^0, e^0), where $e^0 = \mathcal{E}^0 - \rho^0 V$ (or e^0 is the leading term of e_f when expanded in the power series of ε). But, since E is E^0 with $\varphi_1 + \varphi_2 V$ and φ_2 being replaced by $-\mu/\theta$ and $1/\theta$, the moments of $\gamma(E)$ corresponding to A_n are calculated as

$$\int_{\mathbb{R}^3} |v|^{2n} \gamma(E)\, dv = 2^{n+\frac{5}{2}} \pi C B\left(n + \frac{3}{2}, k - n - \frac{3}{2}\right) (-\mu/\theta)^{n-k+\frac{3}{2}} \theta^{n+\frac{3}{2}}.$$

Thus, we have the same relation between (μ, θ) and (ρ_f, e_f) as that between (μ^0, θ^0) and (ρ^0, e^0), because the left-hand side with $n = 0$ and 1 is no other than ρ_f and $2e_f$.[4] In this way, we obtain explicit relation between (μ, θ) and (ρ_f, e_f)

$$\rho_f = 4\sqrt{2}\pi C B\left(\frac{3}{2}, k - \frac{3}{2}\right) (-\mu)^{-k+\frac{3}{2}} \theta^k,$$

$$e_f = 4\sqrt{2}\pi C B\left(\frac{5}{2}, k - \frac{5}{2}\right) (-\mu)^{-k+\frac{5}{2}} \theta^k,$$

or its inverse

$$\mu(\rho_f, e_f) = -\frac{2}{3}\left(k - \frac{5}{2}\right) \frac{e_f}{\rho_f},$$

$$\theta(\rho_f, e_f) = \left(\frac{\left(\frac{2}{3}\left(k - \frac{5}{2}\right)\right)^{k-\frac{5}{2}}}{4\sqrt{2}\pi C B\left(\frac{5}{2}, k - \frac{5}{2}\right)} \left(\frac{e_f}{\rho_f}\right)^{k-\frac{5}{2}} e_f \right)^{\frac{1}{k}}.$$

Note that the chemical potential μ is negative. Further, in the present example, we can obtain explicit expressions for a_1/θ^0, $\bar{a}_1/\theta^0$, and $\bar{A}_1$ in terms of ρ^0 and $\mathcal{E}^0$ as

$$\frac{a_1}{\theta^0} = 3\rho^0, \quad \frac{\bar{a}_1}{\theta^0} = 5\mathcal{E}^0 - 2\rho^0 V,$$

[4] The parallel discussion given here is necessary also in the following examples when we show the relation between (μ, θ) and (ρ_f, e_f), though we do not refer to it.

$$\bar{A}_1 = 2(\mathcal{E}^0 - \rho^0 V)\left(V + \frac{5}{3}\frac{k-\frac{5}{2}}{k-\frac{7}{2}}\frac{\mathcal{E}^0 - \rho^0 V}{\rho^0}\right).$$

Therefore, the diffusion equations (11) and (12) are also written explicitly as

$$\partial_t \rho^0 \; - \frac{2}{3}\nabla_x \cdot \left(\nabla_x(\mathcal{E}^0 - \rho^0 V) + \frac{3}{2}\rho^0 \nabla_x V\right) = 0,$$

$$\partial_t \mathcal{E}^0 \; - \frac{2}{3}\nabla_x \cdot \left(\nabla_x(\mathcal{E}^0 - \rho^0 V)\left(V + \frac{5}{3}\frac{k-\frac{5}{2}}{k-\frac{7}{2}}\frac{\mathcal{E}^0 - \rho^0 V}{\rho^0}\right) + \frac{3}{2}\mathcal{E}^0 \nabla_x V\right) = 0.$$

Example 4.2. A cut-off power with positive exponent, $\gamma(E) = C(E_2 - E)_+^k$, with E_2, C, and k being positive constants.

First, the coefficients a_n's ($n = 0, 1, 2, 3$) are calculated as

$$\begin{aligned} a_n &:= -\int_{\mathbb{R}^3} |v|^{2n}\dot{\gamma}(E^0)\,dv \\ &= 4\pi Ck \int_0^{\sqrt{\frac{2(E_2-\varphi_1-\varphi_2 V)}{\varphi_2}}} |v|^{2(n+1)}\left(E_2 - \varphi_1 - \varphi_2 V - \frac{1}{2}\varphi_2 |v|^2\right)^{k-1} d|v| \\ &= 2^{n+\frac{5}{2}}\pi Ck(E_2 - \varphi_1 - \varphi_2 V)^{k+n+\frac{1}{2}}\varphi_2^{-n-\frac{3}{2}} \int_0^1 s^{n+\frac{1}{2}}(1-s)^{k-1}ds \\ &= 2^{n+\frac{5}{2}}\pi Ck(E_2 - \varphi_1 - \varphi_2 V)^{k+n+\frac{1}{2}}\varphi_2^{-n-\frac{3}{2}} B\left(n + \frac{3}{2}, k\right). \end{aligned}$$

Here, in the last line, we have used another integral representation of the Beta function:[(9)]

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \Re x > 0, \quad \Re y > 0.$$

With this result and the same identity as in the previous example, the tensors A and D in the diffusion equation (16) are written as

$$\begin{aligned} A_{ij} = \frac{8}{3}\sqrt{2}\pi CB\left(\frac{5}{2}, k+1\right)(E_2 - \varphi_1 - \varphi_2 V)^{k+\frac{1}{2}}\varphi_2^{-\frac{7}{2}}\Bigg[\left(k+\frac{3}{2}\right)\left(k+\frac{5}{2}\right)\varphi_2^2\delta_{i1}\delta_{j1} \\ + \left(k + \frac{5}{2}\right)\left(k\varphi_2 V + \frac{3}{2}(E_2 - \varphi_1)\right)\varphi_2(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) \\ + \left(k(k+1)(\varphi_2 V)^2 + 3k(E_2 - \varphi_1)\varphi_2 V + \frac{15}{4}(E_2 - \varphi_1)^2\right)\delta_{i2}\delta_{j2}\Bigg], \end{aligned}$$

$$D_{ij}^{\ell m} = \frac{16}{5}\sqrt{2}\pi C B\left(\frac{7}{2}, k+1\right)(E_2-\varphi_1-\varphi_2 V)^{k+\frac{3}{2}}\varphi_2^{-\frac{9}{2}}\left[\left(k+\frac{5}{2}\right)\left(k+\frac{7}{2}\right)\varphi_2^2\delta_{i1}\delta_{j1}\right.$$

$$+\left(k+\frac{7}{2}\right)\left(k\varphi_2 V + \frac{5}{2}(E_2-\varphi_1)\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1})$$

$$\left.+\left(k(k+1)(\varphi_2 V)^2 + 5k(E_2-\varphi_1)\varphi_2 V + \frac{35}{4}(E_2-\varphi_1)^2\right)\delta_{i2}\delta_{j2}\right]\delta_{\ell m}.$$

In the same way, A_n's ($n = 0, 1, 2$) can be calculated as

$$A_n = 2^{n+\frac{5}{2}}\pi C B\left(n+\frac{3}{2}, k+1\right)(E_2-\varphi_1-\varphi_2 V)^{n+k+\frac{3}{2}}\varphi_2^{-n-\frac{3}{2}},$$

and, therefore, are related to a_n's as

$$A_n = \frac{\varphi_2}{2n+3}a_{n+1}.$$

Therefore, as in the previous example, we obtain explicit relation between (μ, θ) and (ρ_f, e_f)

$$\rho_f = 4\sqrt{2}\pi C B\left(\frac{3}{2}, k+1\right)(E_2\theta+\mu)^{k+\frac{3}{2}}\theta^{-k},$$

$$e_f = 4\sqrt{2}\pi C B\left(\frac{5}{2}, k+1\right)(E_2\theta+\mu)^{k+\frac{5}{2}}\theta^{-k},$$

and its inverse

$$\mu(\rho_f, e_f) = -E_2\theta(\rho_f, e_f) + \frac{2}{3}\left(k+\frac{5}{2}\right)\frac{e_f}{\rho_f},$$

$$\theta(\rho_f, e_f) = \left[4\sqrt{2}\pi C B\left(\frac{5}{2}, k+1\right)\left(\frac{2}{3}\left(k+\frac{5}{2}\right)\frac{e_f}{\rho_f}\right)^{k+\frac{5}{2}} e_f^{-1}\right]^{\frac{1}{k}}.$$

Further, in the present example, we again obtain the explicit expressions for a_1/θ^0, $\bar{a}_1/\theta^0$, and $\bar{A}_1$ in terms of ρ^0 and $\mathcal{E}^0$ as

$$\frac{a_1}{\theta^0} = 3\rho^0, \quad \frac{\bar{a}_1}{\theta^0} = 5\mathcal{E}^0 - 2\rho^0 V,$$

$$\bar{A}_1 = 2(\mathcal{E}^0 - \rho^0 V)\left(V + \frac{5}{3}\frac{k+\frac{5}{2}}{k+\frac{7}{2}}\frac{\mathcal{E}^0-\rho^0 V}{\rho^0}\right).$$

Therefore, the diffusion equations (11) and (12) are written explicitly as

$$\partial_t \rho^0 \; -\frac{2}{3}\nabla_x \cdot \left(\nabla_x(\mathcal{E}^0 - \rho^0 V) + \frac{3}{2}\rho^0 \nabla_x V \right) = 0,$$

$$\partial_t \mathcal{E}^0 \; -\frac{2}{3}\nabla_x \cdot \left(\nabla_x(\mathcal{E}^0 - \rho^0 V)\left(V + \frac{5}{3}\frac{k+\frac{5}{2}}{k+\frac{7}{2}}\frac{\mathcal{E}^0 - \rho^0 V}{\rho^0} \right) + \frac{3}{2}\mathcal{E}^0 \nabla_x V \right) = 0.$$

Example 4.3. The Maxwellian distribution $\gamma(E) = e^{-E}$.

The coefficients a_n's ($n = 0, 1, 2, 3$) are calculated as

$$\begin{aligned} a_n &:= -\int_{\mathbb{R}^3} |v|^{2n} \dot{\gamma}(E^0)\, dv \left(= \int_{\mathbb{R}^3} |v|^{2n} e^{-E^0} dv = A_n \right) \\ &= 4\pi e^{-\varphi_1 - \varphi_2 V} \int_0^\infty |v|^{2(n+1)} \exp\left(-\frac{\varphi_2}{2}|v|^2 \right) d|v| \\ &= (2\pi)^{3/2}(2n+1)!!\, e^{-\varphi_1-\varphi_2 V} \varphi_2^{-n-\frac{3}{2}}, \end{aligned}$$

and therefore the tensors A and D in the diffusion equation (16) are written as

$$\begin{aligned} A_{ij} &= (2\pi)^{3/2}\, e^{-\varphi_1-\varphi_2 V} \varphi_2^{-\frac{7}{2}} \Bigg[\varphi_2^2 \delta_{i1}\delta_{j1} + \left(\varphi_2 V + \frac{3}{2}\right)\varphi_2(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) \\ &\quad + \left(\left(\varphi_2 V + \frac{3}{2}\right)^2 + \frac{3}{2} \right) \delta_{i2}\delta_{j2} \Bigg], \\ D_{ij}^{\ell m} &= 3(2\pi)^{3/2}\, e^{-\varphi_1-\varphi_2 V} \varphi_2^{-\frac{9}{2}} \Bigg[\varphi_2^2 \delta_{i1}\delta_{j1} + \left(\varphi_2 V + \frac{5}{2}\right)\varphi_2(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) \\ &\quad + \left(\left(\varphi_2 V + \frac{5}{2}\right)^2 + \frac{5}{2} \right) \delta_{i2}\delta_{j2} \Bigg] \delta_{\ell m}. \end{aligned}$$

In the present example, since $a_n = A_n$, we obtain explicit relation between (μ, θ) and (ρ_f, e_f) (see the footnote 4)

$$\rho_f = (2\pi)^{\frac{3}{2}}\, e^{\frac{\mu}{\theta}} \theta^{\frac{3}{2}}, \quad e_f = \frac{3}{2}(2\pi)^{\frac{3}{2}}\, e^{\frac{\mu}{\theta}} \theta^{\frac{5}{2}},$$

and its inverse

$$\mu(\rho_f, e_f) = \theta(\rho_f, e_f) \ln \frac{\rho_f}{[2\pi\theta(\rho_f, e_f)]^{\frac{3}{2}}}, \quad \theta(\rho_f, e_f) = \frac{2}{3}\frac{e_f}{\rho_f}.$$

Further, we can obtain the explicit expressions for a_1/θ^0, $\bar{a}_1/\theta^0$, and $\bar{A}_1$ in terms of ρ^0 and $\mathcal{E}^0$ as

$$\frac{a_1}{\theta^0} = 3\rho^0, \quad \frac{\bar{a}_1}{\theta^0} = 5\mathcal{E}^0 - 2\rho^0 V, \quad \bar{A}_1 = \frac{2}{3}\frac{1}{\rho^0}(\mathcal{E}^0 - \rho^0 V)(5\mathcal{E}^0 - 2\rho^0 V),$$

and consequently the diffusion equations (11) and (12) are written explicitly as

$$\begin{aligned} \partial_t \rho^0 \ & -\frac{2}{3}\nabla_x \cdot \left(\nabla_x(\mathcal{E}^0 - \rho^0 V) + \frac{3}{2}\rho^0 \nabla_x V \right) = 0, \\ \partial_t \mathcal{E}^0 \ & -\frac{2}{3}\nabla_x \cdot \left(\frac{1}{3}\nabla_x \frac{1}{\rho^0}(\mathcal{E}^0 - \rho^0 V)(5\mathcal{E}^0 - 2\rho^0 V) + \frac{3}{2}\mathcal{E}^0 \nabla_x V \right) = 0. \end{aligned}$$

Example 4.4. The Fermi-Dirac distribution $\gamma(E) = (e^E + \eta)^{-1}$ with η being a positive constant.

We start with the calculation of a_n's ($n = 0, 1, 2, 3$).

$$\begin{aligned} a_n &:= -\int_{\mathbb{R}^3} |v|^{2n} \dot{\gamma}(E^0)\, dv \\ &= -\partial_{\varphi_1} \int_{\mathbb{R}^3} |v|^{2n} \gamma(E^0)\, dv (= -\partial_{\varphi_1} A_n(\varphi_1, \varphi_2)) \\ &= -2\pi\eta^{-1} \left(\frac{2}{\varphi_2}\right)^{n+\frac{3}{2}} \partial_{\varphi_1} \int_0^\infty \frac{s^{n+\frac{1}{2}}}{\exp(s + \varphi_1 + \varphi_2 V - \ln\eta) + 1} ds \\ &= 2^{n+\frac{5}{2}} \pi\eta^{-1} \varphi_2^{-n-\frac{3}{2}} \Gamma\left(n + \frac{3}{2}\right) \partial_{\varphi_1} \left(\mathrm{Li}_{n+\frac{3}{2}}(-\eta e^{-\varphi_1 - \varphi_2 V}) \right) \\ &= -2^{n+\frac{5}{2}} \pi\eta^{-1} \Gamma\left(n + \frac{3}{2}\right) \mathrm{Li}_{n+\frac{1}{2}}(-\eta e^{-\varphi_1 - \varphi_2 V}) \varphi_2^{-n-\frac{3}{2}}. \end{aligned}$$

Here Li_k is the polylogarithm function[(10)] (de Jonquiére's function) defined over the unit open disk by

$$\mathrm{Li}_k(z) = \sum_{l=1}^\infty \frac{z^l}{l^k}, \quad |z| < 1, \quad z \in \mathbb{C},$$

and on the whole complex plane by the analytic continuation. We have used the formulas

$$\int_0^\infty \frac{s^t}{\exp(s - \nu) + 1} ds \ = -\Gamma(t+1)\mathrm{Li}_{t+1}(-e^\nu), \quad \text{for } t > 0, \nu \in \mathbb{R}, \quad (21)$$

$$z\frac{d}{dz}\mathrm{Li}_k(z) = \mathrm{Li}_{k-1}(z). \quad (22)$$

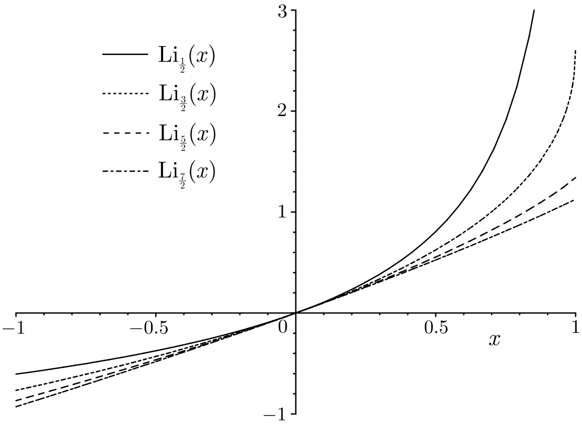

Fig. 1. Polylogarithm function.

The tensors A and D in the diffusion equation (16) are then written as

$$\begin{aligned}
A_{ij} &= -(2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{7}{2}}\Bigg[\mathrm{Li}_{\frac{1}{2}}(-\xi)\varphi_2^2\delta_{i1}\delta_{j1} \\
&\quad + \left(\mathrm{Li}_{\frac{1}{2}}(-\xi)\varphi_2 V + \frac{3}{2}\mathrm{Li}_{\frac{3}{2}}(-\xi)\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}) \\
&\quad + \left(\mathrm{Li}_{\frac{1}{2}}(-\xi)(\varphi_2 V)^2 + 3\mathrm{Li}_{\frac{3}{2}}(-\xi)\varphi_2 V + \frac{15}{4}\mathrm{Li}_{\frac{5}{2}}(-\xi)\right)\delta_{i2}\delta_{j2}\Bigg], \\
D_{ij}^{\ell m} &= -3(2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{9}{2}}\Bigg[\mathrm{Li}_{\frac{3}{2}}(-\xi)\varphi_2^2\delta_{i1}\delta_{j1} \\
&\quad + \left(\mathrm{Li}_{\frac{3}{2}}(-\xi)\varphi_2 V + \frac{5}{2}\mathrm{Li}_{\frac{5}{2}}(-\xi)\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}) \\
&\quad + \left(\mathrm{Li}_{\frac{3}{2}}(-\xi)(\varphi_2 V)^2 + 5\mathrm{Li}_{\frac{5}{2}}(-\xi)\varphi_2 V + \frac{35}{4}\mathrm{Li}_{\frac{7}{2}}(-\xi)\right)\delta_{i2}\delta_{j2}\Bigg]\delta_{\ell m},
\end{aligned}$$

where the notation $\xi = \eta e^{-\varphi_1-\varphi_2 V}$ has been used. Note that the argument $-\xi$ of the polylogarithm functions is negative, and thus all of them occurring above always take a negative value because of the first formula (21) (Fig. 1).

As is obvious from the calculation of a_n's above,

$$A_n = -2^{n+\frac{5}{2}}\pi\eta^{-1}\varphi_2^{-n-\frac{3}{2}}\Gamma\left(n+\frac{3}{2}\right)\mathrm{Li}_{n+\frac{3}{2}}(-\eta e^{-\varphi_1-\varphi_2 V}),$$

and the relation

$$A_n = \frac{\varphi_2}{2n+3} a_{n+1},$$

holds again. Therefore,

$$\rho_f = -(2\pi)^{\frac{3}{2}} \eta^{-1} \theta^{\frac{3}{2}} \mathrm{Li}_{\frac{3}{2}}(-\eta e^{\frac{\mu}{\theta}}), \quad e_f = -\frac{3}{2}(2\pi)^{\frac{3}{2}} \eta^{-1} \theta^{\frac{5}{2}} \mathrm{Li}_{\frac{5}{2}}(-\eta e^{\frac{\mu}{\theta}}),$$

(see the footnote 4). The inversion of this relation can be performed graphically or numerically in the following way. First we transform the above relation into

$$F_{\mathrm{FD}}(\xi) = \eta \left(\frac{3}{4\pi} \frac{\rho_f}{e_f} \right)^{\frac{3}{2}} \rho_f, \quad \theta = \frac{2}{3} \frac{e_f}{\rho_f} R_{\mathrm{FD}}(\xi),$$

where

$$F_{\mathrm{FD}}(s) = -\mathrm{Li}_{\frac{3}{2}}(-s) R_{\mathrm{FD}}(s)^{\frac{3}{2}}, \quad R_{\mathrm{FD}}(s) = \frac{\mathrm{Li}_{\frac{3}{2}}(-s)}{\mathrm{Li}_{\frac{5}{2}}(-s)}, \quad s > 0.$$

Remember that η is merely a given positive constant and that $\mathrm{Li}_{\frac{1}{2}}(-\xi)$, $\mathrm{Li}_{\frac{3}{2}}(-\xi)$, $\mathrm{Li}_{\frac{5}{2}}(-\xi)$ are all negative for $\xi > 0$. It is easy to show, by the use of (22) and the Cauchy–Schwarz inequality, that F_{FD} is monotonically increasing while R_{FD} is monotonically decreasing with ξ (Fig. 2). Therefore the value of ξ is uniquely determined from the first equation for every given set of values of ρ_f and e_f. Once ξ is determined, θ is determined from the second equation. Then μ is obtained from the relation $\mu = \theta \ln(\xi/\eta)$. The above process is summarized as

$$\theta = \frac{2}{3} \frac{e_f}{\rho_f} R_{\mathrm{FD}} \left(F_{\mathrm{FD}}^{-1} \left(\eta \left(\frac{3}{4\pi} \frac{\rho_f}{e_f} \right)^{\frac{3}{2}} \rho_f \right) \right), \; \mu = \theta \ln \left(\frac{1}{\eta} F_{\mathrm{FD}}^{-1} \left(\eta \left(\frac{3}{4\pi} \frac{\rho_f}{e_f} \right)^{\frac{3}{2}} \rho_f \right) \right).$$

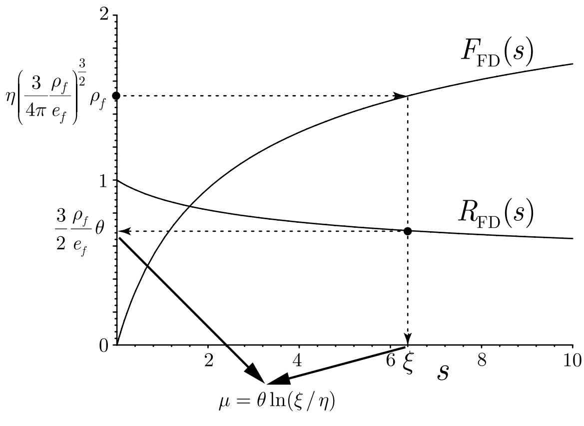

Fig. 2. Monotonic functions $F_{\mathrm{FD}}(\xi)$ and $R_{\mathrm{FD}}(\xi)$ and procedure of inversion.

With the aid of the functions F_{FD} and R_{FD}, the coefficients a_1/θ^0, $\bar{a}_1/\theta^0$, and $\bar{A}_1$ in (11) and (12) are expressed as

$$\frac{a_1}{\theta^0} = 3\rho^0, \quad \frac{\bar{a}_1}{\theta^0} = 5\mathcal{E}^0 - 2\rho^0 V,$$
$$\bar{A}_1 = 2\left[(\mathcal{E}^0 - \rho^0 V)V - \frac{15}{4}(2\pi)^{3/2}\eta^{-1}(\theta^0)^{\frac{7}{2}}\mathrm{Li}_{\frac{7}{2}}(-\xi^0)\right],$$

with

$$\theta^0 = \frac{2}{3}\frac{\mathcal{E}^0 - \rho^0 V}{\rho^0} R_{\mathrm{FD}}(\xi^0), \quad \xi^0 = F_{\mathrm{FD}}^{-1}\left(\eta\left(\frac{3}{4\pi}\frac{\rho^0}{\mathcal{E}^0 - \rho^0 V}\right)^{\frac{3}{2}}\rho^0\right),$$

and the diffusion equations (11) and (12) are written as

$$\partial_t\rho^0 - \frac{2}{3}\nabla_x\cdot\left(\nabla_x(\mathcal{E}^0 - \rho^0 V) + \frac{3}{2}\rho^0\nabla_x V\right) = 0,$$
$$\partial_t\mathcal{E}^0 - \frac{2}{3}\nabla_x\cdot\left(\nabla_x\left[(\mathcal{E}^0 - \rho^0 V)V - \frac{15}{4\eta}(2\pi)^{3/2}(\theta^0)^{\frac{7}{2}}\mathrm{Li}_{\frac{7}{2}}(-\xi^0)\right] + \frac{3}{2}\mathcal{E}^0\nabla_x V\right) = 0.$$

The density ρ^0 and energy $\mathcal{E}^0$ may be obtained from the relations to φ_1 and φ_2 that solve the symmetric diffusion equation (16):

$$\rho^0 = -(2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(-\eta e^{-\varphi_1-\varphi_2 V}),$$
$$\mathcal{E}^0 - \rho^0 V = -\frac{3}{2}(2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{5}{2}}\mathrm{Li}_{\frac{5}{2}}(-\eta e^{-\varphi_1-\varphi_2 V}).$$

Example 4.5. The Bose-Einstein distribution $\gamma(E) = (e^E - \eta)^{-1}$ with η being a positive constant such that $\eta e^{-E} < 1$.[5]

The coefficients a_n's ($n = 0, 1, 2, 3$) are calculated in the same way as the Fermi-Dirac distribution case as

$$\begin{aligned} a_n &:= -\int_{\mathbb{R}^3}|v|^{2n}\dot{\gamma}(E^0)\,dv \\ &= -\partial_{\varphi_1}\int_{\mathbb{R}^3}|v|^{2n}\gamma(E^0)\,dv(= -\partial_{\varphi_1}A_n(\varphi_1,\varphi_2)) \\ &= -2\pi\eta^{-1}\left(\frac{2}{\varphi_2}\right)^{n+\frac{3}{2}}\partial_{\varphi_1}\int_0^\infty\frac{s^{n+\frac{1}{2}}}{\exp(s+\varphi_1+\varphi_2 V - \ln\eta) - 1}ds \end{aligned}$$

[5] Usually, for the Bose–Einstein distribution, η is put unity and $\mu < 0$ is assumed. See, for example, E. A. Jackson, *Equilibrium Statistical Mechanics* (Dover, New York, 2000).

$$= -2^{n+\frac{5}{2}}\pi\eta^{-1}\varphi_2^{-n-\frac{3}{2}}\partial_{\varphi_1}\left(\Gamma\left(n+\frac{3}{2}\right)\mathrm{Li}_{n+\frac{3}{2}}(\eta e^{-\varphi_1-\varphi_2 V})\right)$$

$$= 2^{n+\frac{5}{2}}\pi\eta^{-1}\Gamma\left(n+\frac{3}{2}\right)\mathrm{Li}_{n+\frac{1}{2}}\left(\eta e^{-\varphi_1-\varphi_2 V}\right)\varphi_2^{-n-\frac{3}{2}},$$

where the formulas (22) and

$$\int_0^\infty \frac{s^t}{\exp(s-\nu)-1}ds = \Gamma(t+1)\mathrm{Li}_{t+1}(e^\nu), \quad \text{for } t>0, \nu<0, \tag{23}$$

have been used. Therefore the tensors A and D in the symmetric expression of the diffusion equation (16) are written as

$$\begin{aligned} A_{ij} &= (2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{7}{2}}\Bigg[\mathrm{Li}_{\frac{1}{2}}(\xi)\varphi_2^2\delta_{i1}\delta_{j1} \\ &\quad + \left(\mathrm{Li}_{\frac{1}{2}}(\xi)\varphi_2 V + \frac{3}{2}\mathrm{Li}_{\frac{3}{2}}(\xi)\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}) \\ &\quad + \left(\mathrm{Li}_{\frac{1}{2}}(\xi)(\varphi_2 V)^2 + 3\mathrm{Li}_{\frac{3}{2}}(\xi)\varphi_2 V + \frac{15}{4}\mathrm{Li}_{\frac{5}{2}}(\xi)\right)\delta_{i2}\delta_{j2}\Bigg], \\ D_{ij}^{\ell m} &= 3(2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{9}{2}}\Bigg[\mathrm{Li}_{\frac{3}{2}}(\xi)\varphi_2^2\delta_{i1}\delta_{j1} \\ &\quad + \left(\mathrm{Li}_{\frac{3}{2}}(\xi)\varphi_2 V + \frac{5}{2}\mathrm{Li}_{\frac{5}{2}}(\xi)\right)\varphi_2(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}) \\ &\quad + \left(\mathrm{Li}_{\frac{3}{2}}(\xi)(\varphi_2 V)^2 + 5\mathrm{Li}_{\frac{5}{2}}(\xi)\varphi_2 V + \frac{35}{4}\mathrm{Li}_{\frac{7}{2}}(\xi)\right)\delta_{i2}\delta_{j2}\Bigg]\delta_{\ell m}. \end{aligned}$$

It is easy to see that the argument ξ of the polylogarithms above is in (0, 1), so that they are positive and monotonically increases with ξ.

As is obvious from the derivation of a_n's above,

$$A_n = 2^{n+\frac{5}{2}}\pi\eta^{-1}\Gamma\left(n+\frac{3}{2}\right)\mathrm{Li}_{n+\frac{3}{2}}(\eta e^{-\varphi_1-\varphi_2 V})\varphi_2^{-n-\frac{3}{2}},$$

and the relation

$$A_n = \frac{\varphi_2}{2n+3}a_{n+1},$$

holds again. With these results, we have

$$\rho_f = (2\pi)^{\frac{3}{2}}\eta^{-1}\theta^{\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(\eta e^{\frac{\mu}{\theta}}), \quad e_f = \frac{3}{2}(2\pi)^{\frac{3}{2}}\eta^{-1}\theta^{\frac{5}{2}}\mathrm{Li}_{\frac{5}{2}}(\eta e^{\frac{\mu}{\theta}}),$$

(see the footnote 4). The inversion of this relation can be performed graphically or numerically in the same way as in Example 4.4. We first transform the above

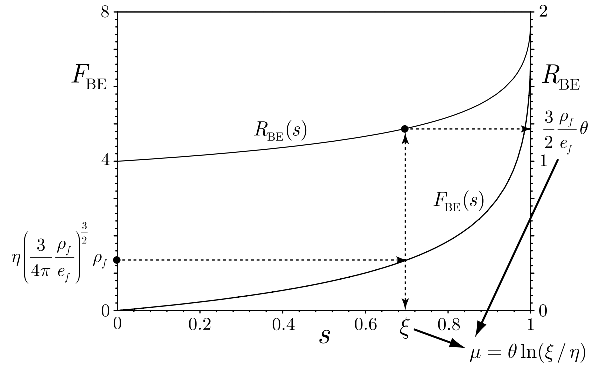

Fig. 3. Monotonic functions $F_{\rm BE}(\xi)$ and $R_{\rm BE}(\xi)$ and procedure of inversion.

relation into

$$F_{\rm BE}(\xi) = \eta\left(\frac{3}{4\pi}\frac{\rho_f}{e_f}\right)^{\frac{3}{2}}\rho_f, \quad \theta = \frac{2}{3}\frac{e_f}{\rho_f}R_{\rm BE}(\xi),$$

where

$$F_{\rm BE}(s) = {\rm Li}_{\frac{3}{2}}(s)R_{\rm BE}(s)^{\frac{3}{2}}, \quad R_{\rm BE}(s) = \frac{{\rm Li}_{\frac{3}{2}}(s)}{{\rm Li}_{\frac{5}{2}}(s)}, \quad 0 < s < 1.$$

The functions $F_{\rm BE}$ and $R_{\rm BE}$ are monotonically increasing with ξ (Fig. 3), which is easily shown by the use of (22) and the Cauchy–Schwarz inequality. Therefore the value of ξ is uniquely determined from the first equation for every given set of values of ρ_f and e_f, and once ξ is determined, θ is determined from the second equation. Then μ is obtained from the relation $\mu = \theta\,\ln(\xi/\eta)$. This process is summarized as

$$\theta = \frac{2}{3}\frac{e_f}{\rho_f}R_{\rm BE}\left(F_{\rm BE}^{-1}\left(\eta\left(\frac{3}{4\pi}\frac{\rho_f}{e_f}\right)^{\frac{3}{2}}\rho_f\right)\right), \mu = \theta\ln\left(\frac{1}{\eta}F_{\rm BE}^{-1}\left(\eta\left(\frac{3}{4\pi}\frac{\rho_f}{e_f}\right)^{\frac{3}{2}}\rho_f\right)\right).$$

Finally, with the aid of $F_{\rm BE}$ and $R_{\rm BE}$, the coefficients a_1/θ^0, $\bar{a}_1/\theta^0$, and $\bar{A}_1$ in (11) and (12) are expressed as

$$\frac{a_1}{\theta^0} = 3\rho^0, \quad \frac{\bar{a}_1}{\theta^0} = 5\mathcal{E}^0 - 2\rho^0 V,$$

$$\bar{A}_1 = 2\left[(\mathcal{E}^0 - \rho^0 V)V + \frac{15}{4}(2\pi)^{\frac{3}{2}}\eta^{-1}(\theta^0)^{\frac{7}{2}}{\rm Li}_{\frac{7}{2}}(\xi^0)\right],$$

with

$$\theta^0 = \frac{2}{3}\frac{\mathcal{E}^0 - \rho^0 V}{\rho^0} R_{\rm BE}(\xi^0), \quad \xi^0 = F_{\rm BE}^{-1}\left(\eta\left(\frac{3}{4\pi}\frac{\rho^0}{\mathcal{E}^0 - \rho^0 V}\right)^{\frac{3}{2}}\rho^0\right),$$

and the diffusion equations (11) and (12) are written as

$$\partial_t \rho^0 \quad -\frac{2}{3}\nabla_x \cdot \left(\nabla_x(\mathcal{E}^0 - \rho^0 V) + \frac{3}{2}\rho^0 \nabla_x V\right) = 0,$$

$$\partial_t \mathcal{E}^0 \quad -\frac{2}{3}\nabla_x \cdot \left(\nabla_x\left[(\mathcal{E}^0 - \rho^0 V)V + \frac{15}{4\eta}(2\pi)^{3/2}(\theta^0)^{\frac{7}{2}}\mathrm{Li}_{\frac{7}{2}}(\xi^0)\right] + \frac{3}{2}\mathcal{E}^0\nabla_x V\right) = 0.$$

The density ρ^0 and the energy $\mathcal{E}^0$ may be obtained from the relations to φ_1 and φ_2 that solve the symmetric diffusion equation (16):

$$\rho^0 = (2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(\eta e^{-\varphi_1 - \varphi_2 V}),$$

$$\mathcal{E}^0 - \rho^0 V = \frac{3}{2}(2\pi)^{\frac{3}{2}}\eta^{-1}\varphi_2^{-\frac{5}{2}}\mathrm{Li}_{\frac{5}{2}}(\eta e^{-\varphi_1 - \varphi_2 V}).$$

5. CONCLUSION

We proposed BGK-type relaxation kinetic models with a general Gibbs state that preserve the mass and energy. The present contribution is along the line of the work by Dolbeault, Markowich, Oelz, and Schmeiser.[(6)] Under the diffusive scaling, we performed formal asymptotic analysis and derived a set of diffusion equations that describes the mass and energy transports. The conservative and symmetric formulations of this set have been presented. We also showed the entropic properties of both the kinetic and diffusion equations.

Finally, as examples of the general Gibbs state, we took the power law with negative exponent, the cut-off power with positive exponent, the Fermi–Dirac, and Bose–Einstein distributions, as well as the conventional Maxwellian, and derived the diffusion equations that could be useful in modeling the transports in porous media, in astrophysics, in semiconductor physics, etc.

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